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ABSTRACT

Counting, naming numbers, numerals, computation, and fractions are the topics covered in this pamphlet. Number lore and interesting number properties are noted; the derivation of some arithmetic terms is briefly discussed. (DT)

NUMBERS AND NUMERALS

By
David Eugene Smith
and
Jekuthiel Ginsburg

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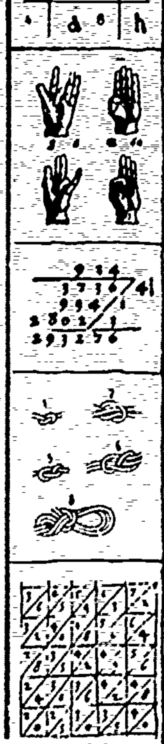

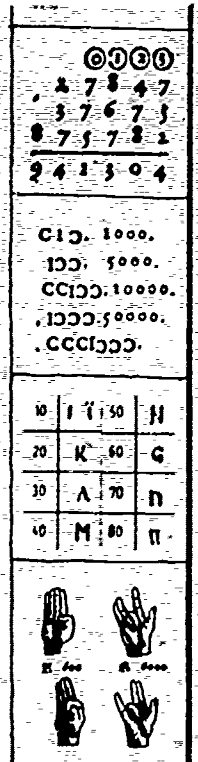

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Editor's Note

This is the first of a contemplated series of inexpensive but important monographs on various phases of mathematics education which is being published by the National Council of Teachers of Mathematics. Other monographs will be published from time to time. It is obvious that these publications will be valuable to teachers of mathematics, and it is also hoped that teachers of the social studies will use them in their classes as supplementary reading material.

My thanks as well as those of the members of the National Council are due to Professors Smith and Ginsburg, without whose interest and devotion this monograph would not have been possible.

W. D. Reeve

Preface But Worth Reading

This is a story of numbers, telling how numbers came into use, and what the first crude numerals, or number symbols, meant in the days when the world was young. It tells where our modern system of numbers came from, how these numbers came to be used by us, and why they are not used everywhere in the world. It tells us why we have ten "figures," whereas one of the greatest nations of the world at one time used only five figures for numbers below five hundred, and another nation used only three.

The story will take us to other countries and will tell us about their numbers and their numerals. The principal lands we shall visit are shown on the map facing page 1. When you read about India or Egypt or Iran (Persia) or Iraq, you can look up that particular country on the map.

The story will tell you how some of the numerals of today came to have their present shape, and how the counters in our shops are related to the numerals which you use, and why some numbers are called "whole" and others "fractional," and why we say "three" cheers instead of "two" or "four" cheers. You will find, too, how people wrote numbers before paper was invented, and how such long words as "multiplication" came into use in Great Britain and America, and how it happened that all of us have to stop and think which number in subtraction is the "minuend" and which the "subtrahend," as if it made any difference to you or to us or to anyone else.

We shall find something interesting about the theory which is nowadays so widely advertised under the name of "numerology" —really one of the last absurdities that has come down to us in relation to the numbers of today. We shall see that it arose from one of the Greek and Hebrew ways of writing numbers long before these peoples knew the numerals which we all use today. Our story will show how the idea of number separated itself from the objects counted and thus became an abstract idea. We shall

also see how this abstract idea became more and more real as it came in contact with the needs of everyday life and with the superstitions of the people.

The story tells many interesting things about measures, a few things about such matters as adding and subtracting, and some things about the curious and hopelessly absurd number superstitions.

If you come across a few strange words from strange languages, do not think that you must pronounce them or keep them in mind. You will find that they are explained, that they are part of the story, and that they help to tell you something interesting. They are no worse than such relics of the past as "subtrahend" and "quotient."

In other words, here is a story of the ages. We hope that you will like it and will tell it to others.

The authors wish to express their thanks to Ginn and Company (Boston, Mass.) for permission to use certain illustrations from Smith's *Number Stories of Long Ago* (1919), his *Rara Arithmetica* (1908), and his *History of Mathematics* (1925, especially Volume II, Chapter I), where the subject is treated more extensively.

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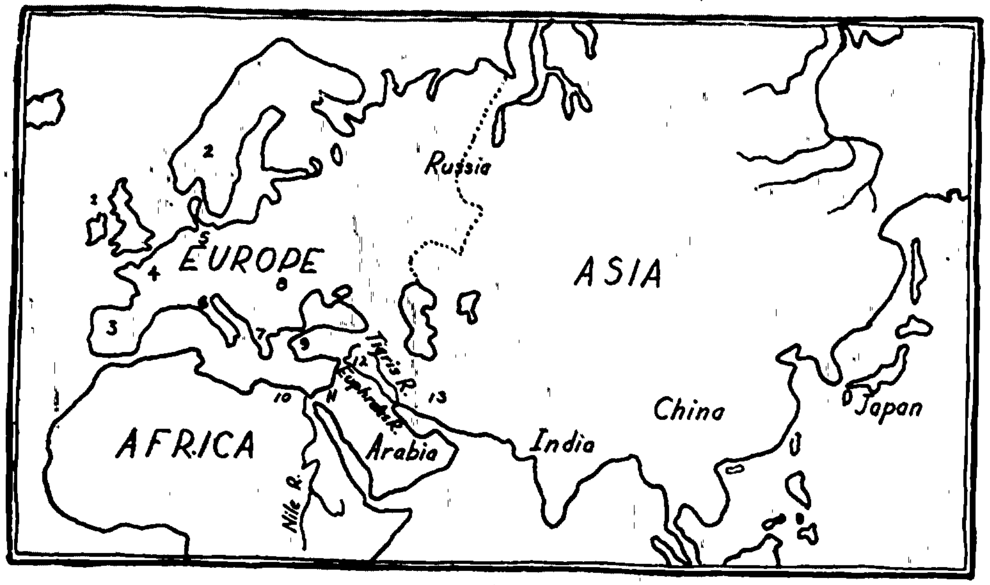


1. British Isles
2. Scandinavia
3. Spain and Portugal
4. France and Netherlands
5. Germany
6. Italy
7. Greece
8. Balkan States
9. Turkey
10. Egypt
11. Syria and Palestine
12. Iraq (Mesopotamia)
13. Iran (Persia)

I. Learning to Count

If you wished to buy a chicken from some poor savage tribe—and such tribes are still to be found in certain parts of Africa, Australia, and some of the Pacific islands—you might point toward some chickens and then hold up one finger. Or, instead of this, you might put one pebble or one stick on the ground. At the same time you might make a sound in your throat, something like *ung*, and the savages would understand that you wanted to buy one chicken.

But suppose you wanted to buy two chickens from these savages; or suppose you wanted three bananas, what would you do?

It would not be hard to make a sign for the number "two." You could show two fingers (held upright or level); or you could point to two shoes, or to two pebbles, or to two sticks. If you knew the word meaning "eyes" in the savages' language, you could point to the chickens and say "eyes" and the savages would know that you wanted two.

For "three" you could not point to eyes or ears or feet and expect the savages to know what number you wished; but you could use three fingers or three pebbles or three sticks. You see that even though you and the savages could not talk to one another, you could easily make the numbers one, two, and three known. It is a curious fact that much of the story of the world begins right here.

Did you ever try to imagine what the world would be like if no one had ever learned how to count or how to write numerals? We are so in the habit of using numbers that we rarely think of how important they are to us.

For example, when we open our eyes in the morning we are likely, first of all, to look at the clock, to see whether it is time to get up, but if people had never learned to count there would be no clocks. We should know nothing of hours or minutes or seconds. We could tell time only by the position of the sun or the

moon in the sky; we could not know the exact time under the best conditions, and in stormy weather we could only guess whether it was morning or noon or night.

The clothes we wear, the houses we live in, and the food we eat, all would be different if people had not learned how to use numbers. We dress in the morning without stopping to think that the materials of which our clothing is made have been woven on machines adjusted to a fraction of an inch. The number and height and width of the stair steps on which we walk down to breakfast were carefully calculated before the house was built. In preparing breakfast we measure so many cups of cereal to so many cups of water; we count the minutes it takes to boil the eggs, or to make the coffee. When we leave the house we take money for bus or car fare, unless we walk, and for lunch, unless we take it with us; but if people could not count there would be no money. All day long we either use numbers ourselves, or we use things that other people have made by using numbers.

It has taken thousands of years for people to learn how to use numbers or the written figures which we call "numerals." For a long time after men began to be civilized such simple numbers as two and three were all they needed. For larger numbers they used words in their various languages which corresponded to such expressions of our own as "lots" of people, a "heap" of apples, a "school" of fish, and a "flock" of sheep. For example, a study of thirty Australian languages showed no number above four, and in many of these languages there were number names for only one and two, the larger numbers being expressed simply as "much" or "many."

You may have heard the numerals, or number figures, called "digits." The Latin word *digiti* means "fingers." Because we have five fingers on each hand, people began, after many centuries, to count by fives. Later they found it more convenient to count by tens, using the fingers of both hands. We still use this "scale" in counting; that is, we count to ten; then to ten tens; then to ten times ten tens; and so on.

Among the Australian tribes in the study mentioned above, the

Andamans, an oceanic tribe, counted to ten with only two words —which you need not learn nor try to read aloud—*ubatúl* (one) and *ikpór* (two). They tapped with the finger tips of either hand, beginning with the first two fingers. They then repeated this gesture with the next two fingers, saying "*anká*" ("and this"). When both hands were finished, the two were brought together to signify $5 + 5$, or 10, and the word "*ardúra*" ("all") was spoken.

In early days people often counted on a scale of three or four instead of ten, and sometimes other number scales were used. Because we have ten toes as well as ten fingers, some people counted fingers and toes together and used a number scale of twenty. In at least one tribe the people said "man finished" for this number. The French in early times counted by twenty (*vingt*). Even now they say "four twenties" ("*quatre vingt*") for eighty, and "four twenties and ten" ("*quatre vingt dix*") for ninety; it was not long ago that they went as far as "nineteen twenties" for three hundred eighty. In the English language, also, this plan was used for a long time; as when we said, "The days of a man's life are three score years and ten," the word "score" meaning twenty.

There are many evidences that twelve was often used as a scale in counting: as 12 inches = 1 foot; 12 ounces = 1 pound (old style); 12 pence = 1 shilling; 12 units = 1 dozen; and 12 lines = 1 inch. There are certain advantages in using a scale of twelve; because $\frac{1}{3}$ of 12 pence = 4 pence, whereas $\frac{1}{3}$ of 10 cents = 3.3333+ cents—a difficult number to work with.

Whatever scale we use, we need as many digits as the scale contains. For example, on our scale of ten we need ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. If we used the scale of eight, we should need 0, 1, 2, 3, 4, 5, 6, 7, and we would write eight as 10 (1 eight and no units); nine would be 11 (1 eight and 1 unit); sixteen would be 20 (2 eights and no units); and so on. If we were brought up with such a system it would be just as easy as our scale of ten; in some respects it would be even easier because eight is more easily divided into fourths and eighths (fractions that are often needed) than ten.

II. Naming the Numbers

NUMBER names were among the first words used when people began to talk. They were needed in speaking of days, sheep, men, birds, and all sorts of things that people had to deal with in their everyday lives.

Some of these words have probably not changed a great deal in many thousands of years. You remember the throaty grunt "*ung*" which meant "one" to the savage who was selling chickens. It is not hard to see how it developed into the words meaning "one" in the following list. This list also shows how nearly alike some of the other number names are in several different languages. The blank spaces mean that those number names were not like the ones in the list.

Modern English	French	German	Old English	Latin	Greek
one	un	ein	an	unus	oinos
two	deux	zwei	two	duo	duo
three	trois	drei	threo	tres	treis
four		vier	feower		
five		fünf	fif		
six	six	sechs		sex	hex
seven	sept	sieben	seofon	septem	hepta
eight	huit	acht	ahta	octo	okto
nine	neuf	neun	nigon	novem	ennes
ten		zehn	tien	decem	deka

In the case of foreign words like these, do not try to remember them or to read them aloud.

How did it come about that these words are so much alike in all these different languages?

Find India on the map facing page 1. In ancient days the people who lived in this country spoke a language known as Sanskrit. Some of these people traveled west to Greece and Italy,

and to other European lands farther north and west. The languages spoken by the people who lived in these countries later are called by different names, Latin, Greek, German, and so on; but they are grouped together in a class known as Indo-European languages, because the very beginning of many of the words in each language came from the ancient Sanskrit. For instance, the Sanskrit word for seven was *sapta*. You can see by looking at the list how the other names for seven have grown out of this Sanskrit word. If you are interested in the travels of the number names, you can find out a great deal about their history in any large dictionary.

The number "one" was looked upon by some ancient peoples as being different from the other numbers. In the old Hebrew and Arabic languages counting began with two, one being reserved for God alone, as in the expression: "There is one God," or "God is One." The Arabic word for one was *u-khd*, the Hebrew word, *akh-d*, and the Syrian word, *kh-ad*.

Even in Europe in the early days, the zero (0) and the one (1) were not called digits, because the zero was not considered a number and the one was looked upon as the source of all numbers and not as a number itself. Writers of that time applied the name "digits" only to the eight numerals from 2 to 9. Nowadays, however, the first ten numerals, beginning with 0, are usually called the digits.

In some languages two also was considered different from the other numbers. The Greeks had a special way of showing that they were speaking of two things. For example, our word "citizen" is, in Greek, *polites*; two citizens would be *polita*, and any higher number of citizens, *politai*, the last syllable showing whether there are two or more. We find the same usage in Hebrew, the plural (more than two) ending in *im*, and the dual (two of the objects) ending in *aim*. We should be thankful that our own language is not unnecessarily burdened in this way.

The numbers used in counting, such as one, two, three, are called "cardinals." The adjectives that show the order of the objects counted, such as first, second, third, are called "ordinals."

It is a curious fact that in many languages the words meaning "first" and "second" do not come from the words meaning "one" and "two." Thus we have:

English	one	first
German	ein	der erste
Latin	unus	primus
Russian	od-in	pervi
Hebrew	akh-d	rishon
French	un	premier
Italian	uno	primo

Also, it is easy to see that the word "second" is not related to the word "two." All this goes to show that early people did not connect "the second boy" with "two boys." It was not until the human race had developed considerably that people began to see this relationship and to relate "third" to "three," and "fourth" to "four."

If you look at the lists of number names on page 4 you will see that many other words have grown out of these names. The last four words in the Latin column will remind you of the names of the last four months in our year. You may wonder why the name of our ninth month (September) begins with *septem* (seven) our tenth month (October) with *octo* (eight), and so on with November and December. The reason is that the calendar year could, of course, begin with any month that people might choose. In the Latin world, which included about all of Western Europe at the time the present months were named, the year was often taken as beginning in March. When this was done, the seventh month was September, and so on to the shortest month of the year, which was the tenth, December.

III. From Numbers to Numerals

WE DO not know how long ago it was that human beings first began to make their thoughts known to one another by means of speech; but it seems probable that people learned to use words in talking many thousands of years before they learned to set down these words in writing. In the same way, after people learned to name numbers it took a long time for them to learn to use signs for the numbers; for example, to use the numeral "2" instead of the word "two."

Where and when did the use of numerals begin? This question takes us back to the very beginnings of history. If you look at the map you will see Egypt, lying along the valley of the Nile. As long ago as 3000 B. C.—perhaps even earlier—there were prosperous cities in Egypt with markets and business houses, and with an established government over all the land. The keeping of the commercial and government records necessitated the use of large numbers. So the Egyptians made up a set of numerals by which they could express numbers of different values from units up to hundreds of thousands.

In another valley, between the Tigris and Euphrates Rivers in Asia, there lies a part of the territory now known as Iraq. This country was formerly called Mesopotamia, or "the land between the rivers." Five thousand years ago the Sumerians, a strange people who seem to have come from the mountainous regions of Iran (Persia), had developed a high degree of civilization in this land. They could read and write and had a usable system of numbers and numerals. It will help us to realize what a long, long time ago this was if we recall that Abraham, the Hebrew patriarch, lived as a boy in this same country "between the rivers," a thousand years later than this; that is, about 2000 B. C. By the time that Abraham was born another race of people, the Babylonians, were taking the power away from the Sumerians. Meanwhile, however, the Babylonians were learning from the

Sumerians how they carried on trade with other nations, and how they built their houses with bricks of baked clay, and wrote their letters and their historical records on the same sort of bricks. They also learned how to use the number symbols which the Sumerians seem to have invented. Inscriptions showing how both Sumerians and Babylonians kept their accounts are still in existence. In India, also, numerals were used in ancient days; and the Greeks and Romans, too, had their own ways of making number symbols.

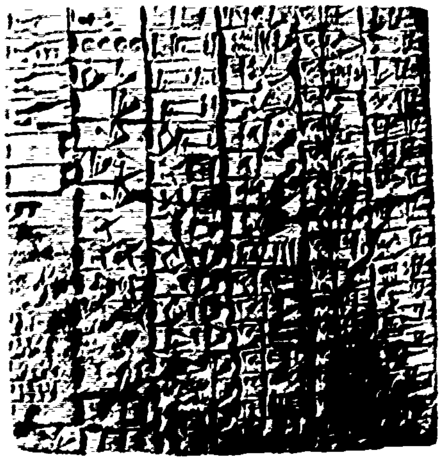
To express the number "one" all these ancient peoples at times made use of a numeral like our "1." The Egyptians painted it on pottery and cut it on stone at least five thousand years ago; and somewhat later the Sumerians taught the Babylonians to stamp it on their clay tablets. More than two thousand years ago the Greeks and the Romans used this symbol, and in most of India the natives did the same. This numeral probably came from the lifted finger, which seems to be the most easy and natural way of showing that we mean "one."

But you will remember that we found several different ways of showing the savage poultrymen that we wanted to buy one chicken. We could lift one finger, or we could lay down one pebble or one stick on the ground. In the same way, the "one" was sometimes expressed by a symbol representing a pebble or a bead, and sometimes by a line like this —, representing a stick laid down.

The "two" was commonly expressed by two fingers || or two lines =. If you write || rapidly you will have N, from which has come the figure ۲ used by the Arabs and Persians. The = written rapidly becomes Z, which developed into our 2. The symbol = is used in China and Japan today. The people of India have many languages and various ways of writing numbers, but generally in early times they preferred the = to the ||, so that our 2 was the favorite in many eastern countries, although the Arabic and Iranian (Persian) figure is even today almost universally used in the Mohammedan lands.

Cuneiform, or wedge-shaped, writing was much in use in ancient times. This developed in the Mesopotamian valley, where

the lack of other writing materials led the people to stamp inscriptions on clay bricks with sticks which usually were triangular with sharp edges. The cuneiform numerals for 1, 2, and 3 are 𒁹 𒈫 𒐈. These numerals first appear on the clay bricks of the Sumerians and the Chaldeans, but they were used afterwards by the Babylonians, the Hittites, the Assyrians, and other ancient races. They have been found as far west as Egypt, as far north as Asia Minor (Anatolia), and as far east as Iran (Persia). They are known to have been used about five thousand years ago, and to have continued in use for about three thousand years.



Sumerian Tablet in the State Museum, Berlin. In the second column, line 2, is the number 60 + 10 + 10 + 10 + 10, or 100. (From Menninger, *Zahlwort und Ziffer*.)

In writing numerals the Babylonians sometimes used a stick with a circular cross section, which gave the "one" the shape of a pebble or a bead. They thus had two types of numerals, as may be understood from the following table:

Triangle	𒁹	𒌋	𒁹	𒁹𒌋
Circle	D	O	D	DO
Value	1	10	60	60 × 10

The "three" was commonly represented by three lines, III, or more commonly in the Far East, ≡. In China and Japan, even today, the most common form is 三. When people began to write with some kind of pen they joined these lines together in a form like this: 3 ; and from this came our 3. In the same way, when they came to write III rapidly they used forms like ٣, which finally became the Iranian (Persian) and Arabic ٣, used by millions of people today. It is interesting to see that when this is turned over on its side we have ३, and this accounts for the shape of the three in many East Indian regions where we find the Sanskrit ३, and many similar forms.

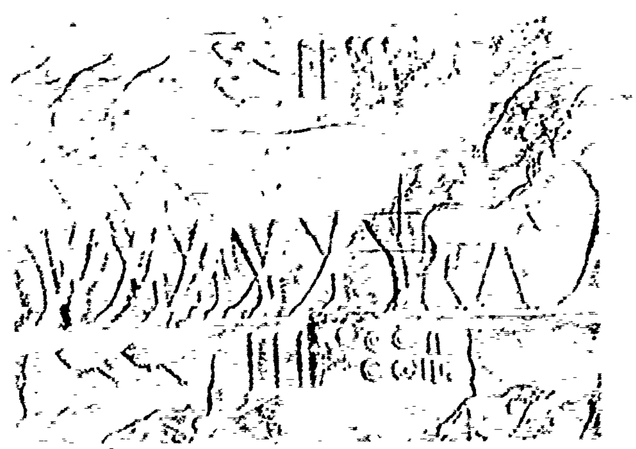
Since in most countries the form of the numerals meaning one, two, and three developed from the arrangement of sticks I II III or 一 二 三, we might naturally expect four to be written in the same way. It is true that this was done in Egypt and Rome (IIII), in Babylonia (𒁹𒁹𒁹𒁹), and in a few other countries; but the lines were generally not joined together. Of all the ancient numerals signifying four, only the Arabic Σ was made by joining four lines. This Arabic numeral appears today as ٤.

Combining numerals and writing large numbers presented a different sort of problem. Five thousand or more years ago the Egyptians used numerical symbols like these:

The Number 27,529 as Written by the Early Egyptians

In the above a bent line is used to represent tens of thousands. But in the next inscription, cut in stone, the line is straight and the symbol for hundreds (in the bottom row of numerals) is curved somewhat differently from the one used in the number given above.

Among the oldest systems of numerals are those used by the Chinese, and later adopted by the Japanese. These have naturally changed somewhat in the course of the centuries, but it is necessary to mention only two types. The first is based on the



This inscription is from a monument now in Cairo, and is part of an inventory of the king's herds of cattle. The number at the top is 223,000, and the one at the bottom is 232,413. (From Menninger, *Zahlwort und Ziffer.*)

use of sticks laid upon a table and used for the purposes of calculating, but it was also used in written documents. It is as follows:

𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
1	2	3	4	5	6	7	8	9
𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱
10	20	30	40	50	60	70	80	90

The system commonly used to represent the first ten numbers is as follows:

一	二	三	四	五
1	2	3	4	5
六	七	八	九	十
6	7	8	9	10

The Greeks had several ways of writing their numbers, but we shall consider only two of them. In one method they simply used the initial letters of the number names, but their letters were quite different from ours. For example, the following early letters and names were used:

Number	Name	Letter
1000	kilo or chilo	X, our *ch*
100	hekto	H, early form
10	deka	Δ, our *d*
5	penta	Π, or Γ, our *p*

These were often combined as shown below, the number being 2977

XX𐅅	HHHH	𐅄	ΔΔΓII
2000 500	400	50	27

The later Greeks, about two thousand years ago, generally used the first ten letters of their alphabet to represent the first ten numbers. For larger numbers they used other letters: K′ for 20, Λ′ for 30, and so on. They often placed a mark (/ or ′) by each letter to show that it stood for a number. The letters, representing numerals from 1 to 9, were then as follows:

A′	B′	Γ′	Δ′	E′	F′	Z′	H′	Θ′
1	2	3	4	5	6	7	8	9

The tens were the next nine letters:

I′	K′	Λ′	M′	N′	Ξ′	O′	Π′	Q′
10	20	30	40	50	60	70	80	90

The hundreds were the next nine letters:

P′	Σ′	T′	Υ′	Φ′	X′	Ψ′	Ω′	Z′
100	200	300	400	500	600	700	800	900

(Q and Z are here used in place of two ancient Greek letters not in our alphabet.)

One of the most interesting evidences of the early use of the initial Greek numerals is seen in a vase now in the Museum at Naples. The picture refers to the Persian wars of the time of Darius, about 500 B. C. In the lower row of figures there is, at the left, a man seated at a table and holding a wax diptych (two-winged) tablet on which are letters which represent 100 *tálenta* (talents of money). On the table are the letters MXHΔΠO<T. The first five of these letters represent the number of talents: 10,000, 1000, 100, 10, and 5. The last three letters stand for 1 *obol* (a fraction of a talent), $\frac{1}{2}$ *obol*, and $\frac{1}{4}$ *obol*.

The drawing below, which has been enlarged from the figure appearing on the vase, shows the numbers more distinctly. The man seems to be a receiver of taxes or a money changer. There is a possibility that the table was used for computing by counters, as explained in Chapter IV.

We are all familiar with such numerals as I, II, III, and IIII or IV as seen on the faces of clocks. These have come to us through many centuries, having been used by the ancient Romans. We speak of them as "Roman numerals." There are seven such characters which are used at present in numbering

chapters of books, volumes, the main divisions of outlines, and the like. They are our common letters with values as follows:

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

They have changed their shapes from time to time, the following being one of the early forms:

I	V	X	L	C	D	CIƆ ∞
1	5	10	50	100	500	1000

This shows how the number 2,752,899 might have been written in the late Roman form:

$\overline{\text{CIƆCIƆDCCL}}$CIƆCIƆDCCCLXXXXVIIII

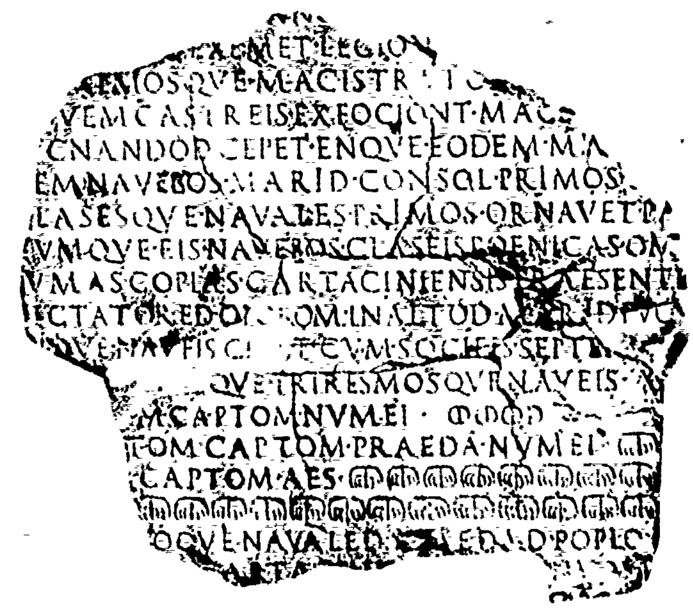
In the earliest inscriptions on stone monuments the "one" was a vertical stroke I, as in the other systems in western Europe. In the Middle Ages, after the small letters became common, i and j were used, the j being usually placed at the end of the number, as in vij for seven.

The earliest and most interesting use of the large Roman numerals is found in a monument set up in Rome to commemorate the victory over the Carthaginians. In the second and third lines from the bottom of the picture on page 15 is the numeral for 100,000, repeated twenty-three times, making 2,300,000. It shows the awkwardness of the Roman numerals as written about twenty-two hundred years ago.

The "five" was generally V, perhaps as representing a hand. This naturally suggests X (two V's) for ten. It is quite as probable, however, that the X came from the crossing off of ten ones. There are various other suggestions in regard to the origin of such Roman numerals. In the Middle Ages the U, which was only another form for V, was used for five, as in uiij for eight.

Fifty was represented in several ways, but L was most commonly used. In the Middle Ages we find many such forms as Mlxj for 1061.

The Roman word for 100 was *centum*, and that for 1000 was *mille*. This probably accounts for the use of C for 100, and M for 1000, although other forms, from which these seem to have been derived, were used in early times.



Numerals on the Columna Rostrata, 260 B. C. In the Palazzo dei Conservatori, in Rome

The Romans often wrote four as IIII, and less often as IV, that is, I from V. On clock faces we find both of these forms even today. It was easier for the ancients to think of "five (fingers) less one" than of "four," and of "ten less one" than of "nine." We show this when we say "15 minutes to 10" instead of "45 minutes after 9," or "a quarter to 3" instead of "three quarters after 2." In writing IV, in which I is taken from V, we use what is called the "subtractive principle." This is found in various systems besides the Roman. The first trace that we find is in the Babylonian clay tablets of 2000 B. C. and earlier. In these there

frequently occurs the word lal () to indicate subtraction. Thus we have*

 for 10 + 10 − 1, or 19

 for 10 + 10 + 10 + 10 − 3, or 37

The following notes relating to the Roman numerals may be interesting:

9 was written IX (that is, I from X) but until the beginning of printing it appeared quite as often in the form VIIII.

19 was written XIX (that is, X + IX), but it also appeared as IXX (that is, 1 from 20).

18 commonly appears as XVIII, but IIXX was also used.

CIↃ was a favorite way of writing 1000, but was later changed to M, the initial of *mille*, 1000. Half of this symbol, either CI or IↃ, led to the use of D for 500.

In writing larger numbers the Romans made use of the following forms:

CIↃ	for 1000
CCIↃↃ	for 10,000
CCCIↃↃↃ	for 100,000

One of the most prominent arithmetics published in England in the sixteenth century (Baker's 1568) gives this curious way of writing 451,234,678,567:

four CliM, two Cxxxiiii, millions, sixe ClxxviiiM, five Clxvii.

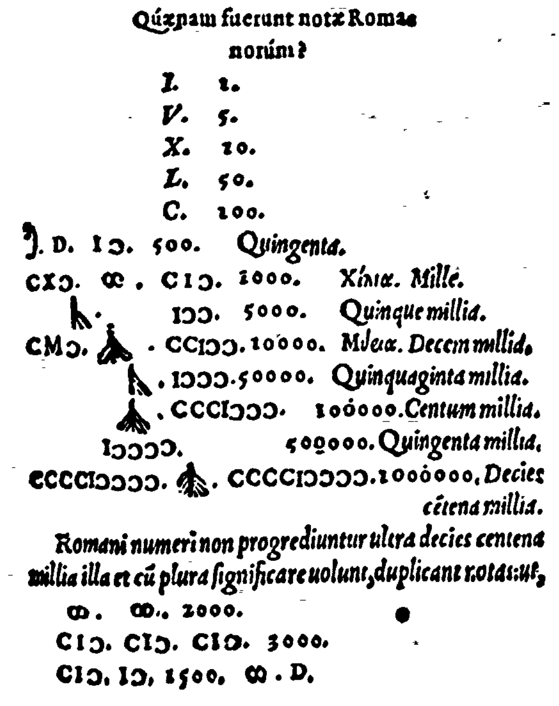
It would seem that this is enough to puzzle both teachers and pupils, but when we remember that our word "billion" ("thousand million" in most European countries) was not in common use in the sixteenth century, we can understand that it means

$$[(4C+51)\,1000\text{'s}+2C+34]\,1000000\text{'s}+(6C+78)\,1000\text{'s}+5C+67$$

and there is no difficulty in seeing that the latter expression means $451{,}234 \times 1000000 + 678 \times 1000 + 567$.

* See O. Neugebauer, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, Bd. I, *Vorgriechische Mathematik*, Berlin, 1934, p. 17.

The following from the work of a Swiss scholar, Freigius, published in 1582, shows the forms of the Roman numerals recognized in his time.


Quænam fuerunt notæ Romanorum?

I. 1.
V. 5.
X. 10.
L. 50.
C. 100.
D. IↃ. 500. *Quingenta.*
CXↃ. ∞. CIↃ. 1000. *Χίλια. Mille.*
IↃↃ. 5000. *Quinque millia.*
CMↃ. CCIↃↃ. 10000. *Μύρια. Decem millia.*
IↃↃↃ. 50000. *Quinquaginta millia.*
CCCIↃↃↃ. 100000. *Centum millia.*
IↃↃↃↃ. 500000. *Quingenta millia.*
CCCCIↃↃↃↃ. CCCCIↃↃↃↃ. 1000000. *Decies cētena millia.*

Romani numeri non progrediuntur ultra decies centena millia illa et cū plura significare uolunt, duplicant notas: ut,

∞. ∞. 2000.
CIↃ. CIↃ. CIↃ. 3000.
CIↃ. IↃ, 1500. ∞. D.

The Roman numerals were commonly used in bookkeeping in European countries until the eighteenth century, although our modern numerals were generally known in Europe at least as early as the year 1000. In 1300 the use of our numerals was forbidden in the banks of certain European cities, and in commercial documents. The argument was that they were more easily forged or falsified than Roman numerals; since, for example, the o could be changed into a 6 or a 9 by a single stroke. When books began to be printed, however, they made rapid progress, although the Roman numerals continued in use in some schools until about 1600, and in commercial bookkeeping for another century.

One reason why the Roman numerals were preferred in bookkeeping was that it is easier to add and subtract with them than with our modern numerals. This may be seen in these two cases:

Addition		Subtraction	
DCCLXXVII	(777)	DCCLXXVII	(777)
CC X VI	(216)	CC X VI	(216)
DCCCCLXXXXIII	(993)	D L X I	(561)

In such work as this it is unnecessary to learn any addition or subtraction facts; simply V and V make X, CC + CC = CCCC, and so on. The only advantage of our numerals in addition and subtraction is that ours are easier to write. As to multiplication and division, however, our numerals are far superior. The ancient Romans used to perform these operations by the use of counters such as those mentioned in Chapter IV.

The Hebrews used their alphabet in writing numerals in the same way as the Greeks; that is, the first ten letters represented the first ten numbers, as shown below.

א ב ג ד ה ו ז ח ט י

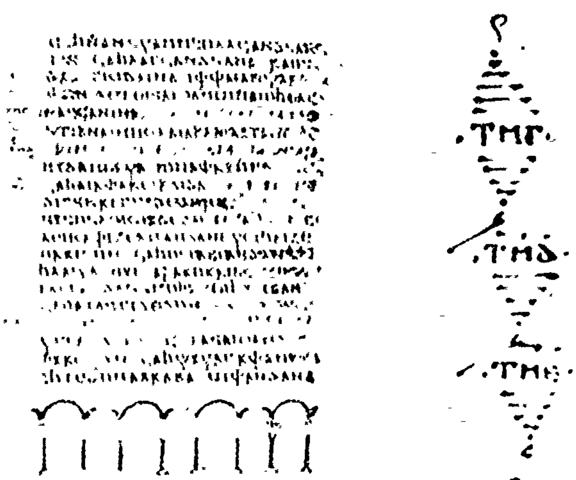
10 9 8 7 6 5 4 3 2 1

The letters and numerals here shown are arranged from right to left, this being the way of writing used by the Hebrews.

Another interesting set of alphabetic numerals was used by the Goths, a people first known in Poland and Germany, who later conquered a considerable part of Europe. These numerals are for the most part of Greek origin. They are shown in the illustration at the top of the next page. Just below to the left is a page from a Bible translated into Gothic by Bishop Ulfilas in the fourth century. On its left margin is a row of numerals. These numerals, enlarged, are shown in the small picture to the right. If you compare these with the alphabetic numerals in the picture above you can easily read them. The first is 300 + 40 + 3, or 343. The second is 344, and the third is 345.

1	A	10	I Ï	100	R
2	B	20	K	200	S
3	Γ	30	λ	300	T
4	d	40	M	400	Y
5	Є	50	N	500	F
6	u	60	G	600	X
7	Z	70	n	700	Θ
8	h	80	Π	800	Ω
9	Φ	90	Ч	900	↑

Alphabetic numerals used by the Goths. (In part from Menninger, *Zahlwort und Ziffer.*)



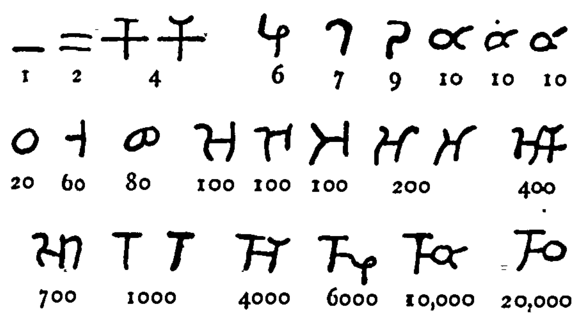
Page from the *Codex Argenteus* (Silver Manuscript), a Bible translated into Gothic by Bishop Ulfilas in the fourth century. (From Menninger, *Zahlwort und Ziffer.*)

We now come to the numerals that are used in Europe and the Americas today, as well as in certain parts of Asia and Africa and regions such as Australasia which were settled by Europeans. First of all, it is necessary to understand that although our European and American numerals are often spoken of as Arabic, they have never been used by the Arabs. They came to us by means of a book on arithmetic which apparently was written in India about twelve hundred years ago, and was translated into Arabic soon afterward. By chance this book was carried by merchants to Europe and there was translated from the Arabic into Latin. This was hundreds of years before books were first printed in Europe, and this arithmetic book was known only in manuscript form. Since it had been translated from Arabic, the numerals were supposed to be those used by the Arabs, but this was not the case. They might be called Hindu-Arabic, but since they took their present shapes in Europe they may better be called European or Modern numerals.

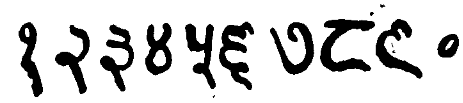
We have seen that our modern numerals 1, 2, and 3 have a long history. The four as we make it, 4, is not as old as these. It is first found in fairly common use in Europe about seven hundred years ago. Among the forms for four in common use in India two thousand years ago were $\times$ and $+$, but there were and still are many other forms used in Asia. The origin of the rest of the numerals is generally unknown. Since in most countries in early days the priests were practically the only educated persons, and since travel was so difficult that different tribes developed different languages, the priests simply invented their own letters and numerals. As travel became easier and as merchants and rulers felt the need for writing, the numerals of the various tribes tended to become more alike. Today, international trade has made the European numerals quite generally known all over the world, although the Chinese or the Arabic forms are still the ones most commonly used by many millions of people.

Our present numerals have changed a great deal from their original forms. Following are some early Hindu characters, found in a cave in India and dating from the second or third century

B. C. Some of these are certainly numerals, and the others probably are.



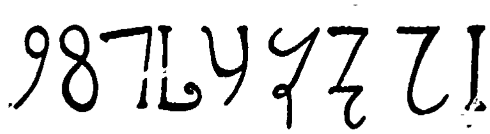
The Hindu numerals which are in some cases closely related to our [illegible] which were taken to Baghdad, in Iraq, about a thousand years ago, are as follows:



The Arabic numerals used then and now are as follows:

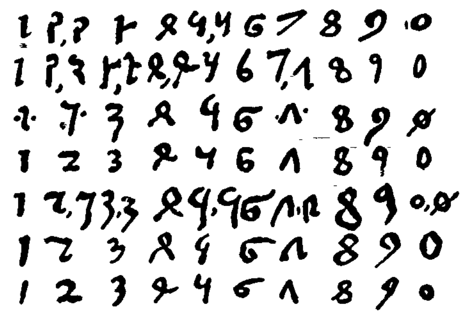
You will observe that the zero is simply a dot and that the five is quite like our zero. The Iranian (Persian) numerals are substantially the same except for the four and the five.

Coming now more closely to our modern numerals we have here the oldest example of all these forms (lacking zero) known in any European manuscript. This was written in Spain in the year 976.



The following table shows the changes in our numerals from the time of their first use in Europe to the beginning of printing.

1 2 3 4 5 6 7 8 9 0



Twelfth century
1197 A.D.
1275 A.D.
c. 1294 A.D.
c. 1303 A.D.
c. 1360 A.D.
c. 1442 A.D.

After they began to appear in books there were few changes of any significance, the chief ones being in the numerals for four and five.* Even at the present time the forms of our numerals frequently change in the attempt to find which kind is the most easily read. For example, consider the following specimens and decide which is the easiest for you to read:

1234567890 1234567890 1234567890

The names for large numbers have also changed from time to time. For example, the word *million* seems not to have been used before the thirteenth century. It means "a big thousand," *mille* being the Latin for thousand, and *-on* meaning (in Italian) *big.* The word started in Italy, was taken over by France in the fifteenth century or earlier, and was thereafter used in England. Until the seventeenth century, people generally spoke of "a thousand thousand" rather than of "a million," and they do so today in certain parts of Europe.

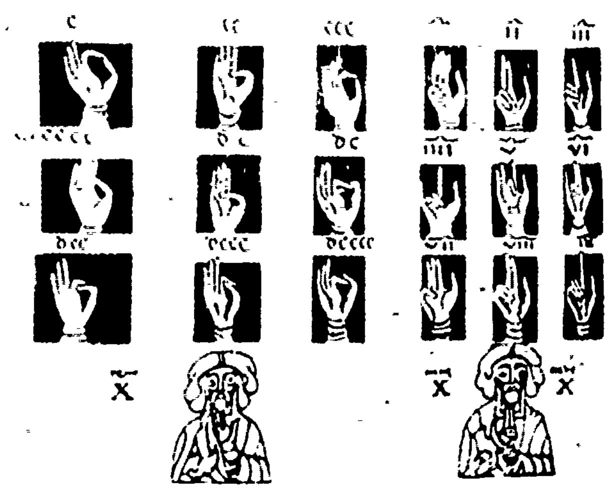
"Billion" is a relatively new word. It comes from the Italian, and is first found as *bimillion*, *bilioni*, and *byllion.* It originally meant a million million, and in England it is generally still so

*See Smith and Karpinski, *The Hindu-Arabic Numerals* (Boston, 1911); Sir G. F. Hill, *The Development of Arabic Numerals in Europe* (Oxford, 1915); D. E. Smith, *History of Mathematics* (2 vol., Boston, 1923, 1925).

understood, although the American use of the term to mean a thousand million has come to be generally understood of late years, largely because of the enormous numbers now in use in financial transactions which concern all countries. The larger numbers have names like trillions, quadrillions, quintillions, and so on, but these are seldom used.

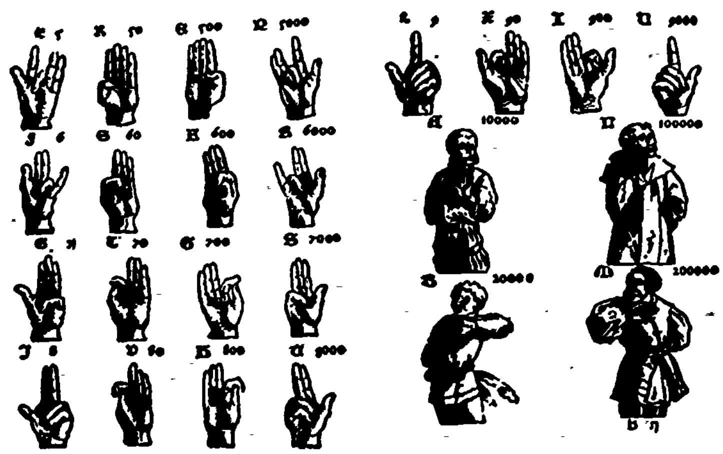
Besides the written numerals which have been described, finger numerals were used during many centuries and by many peoples. The ancient Greeks and Romans used them as did the Europeans of the Middle Ages; and the Asiatics in later times. Even today they are not infrequently used for bargaining in the market places. Indeed, "counting on the fingers" and even multiplying and dividing by these means are known in certain countries, but not commonly in Western Europe and the American continents.

The following illustration shows a few of the finger symbols as they appear in a manuscript written in Spain in the thirteenth century.



Finger numerals. Spanish, Thirteenth Century.
The top row represents 100, 200, 300, 1000. 2000, 3000

The finger symbols below are from a book printed in the sixteenth century, three hundred years after the Spanish manuscript mentioned above. Each represents only part of the system. It

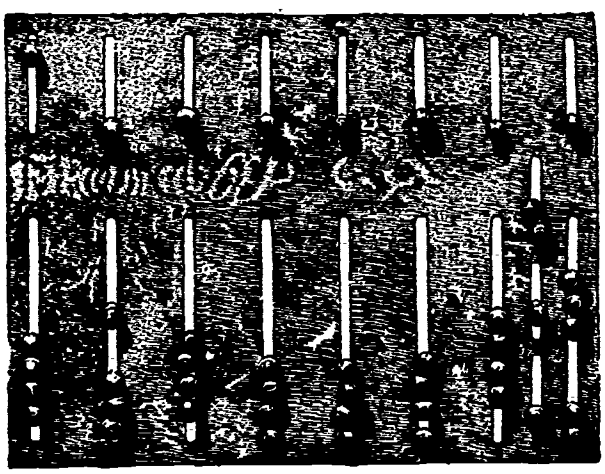


Finger symbols from an arithmetic printed in Germany in 1532

may be observed, however, that the 500 (d) is the same in both illustrations, as are also the 600 (dc), 700 (dcc), 800 (dccc), and 900 (dcccc). There is a similar correspondence in the other numerals not given in these partial illustrations.

IV. From Numerals to Computation

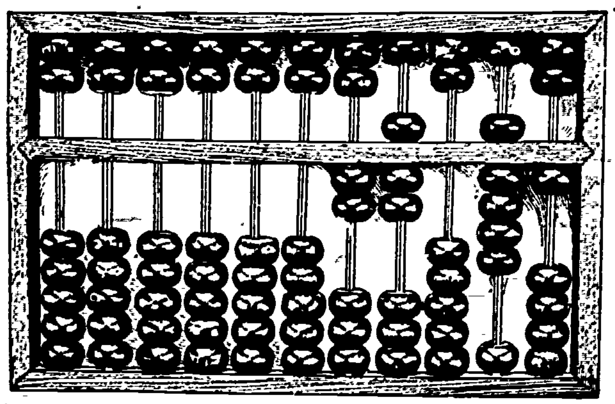
WHEN people first began to use numbers they knew only one way to work with them; that was, to count. Little by little they found out how to add, subtract, and multiply; but this was slow work and in some countries special devices were invented to make computation easier, especially in dealing with large numbers. The Romans used a counting table, or abacus, in which units, fives, tens, and so on were represented by beads which could be moved in grooves, as shown in this illustration.



Ancient bronze abacus used by the Romans

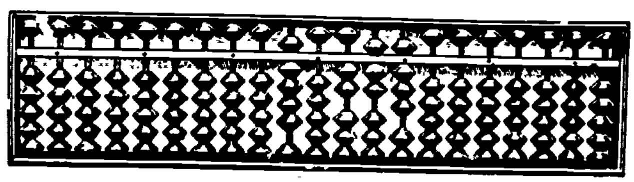
They called these beads *calculi*, which is the plural of *calculus*, or pebble. We see here the origin of our word "calculate." Since the syllable *calc* means lime, and marble is a kind of limestone, we see that a calculus was a small piece of marble, probably much like those used in playing marbles today. Sometimes, as in the Chinese abacus described below, the calculi slid along on rods. This kind of abacus is called a *suan-pan*, and it is used today in all

parts of China. In the one shown in the following illustration the beads are arranged so as to make the number 27,091, each bead at the top representing five of the different orders (units, tens, and so on).



Chinese abacus

The Japanese use a similar instrument known as the *soroban.* The Chinese and Japanese can add and subtract on the abacus, or counting board, much more rapidly than we can with pencil and paper, but they cannot multiply and divide as quickly as we can. In the *soroban* here shown, beginning in the twelfth column from the right, the number represented is 90,278. The other columns are not being used.

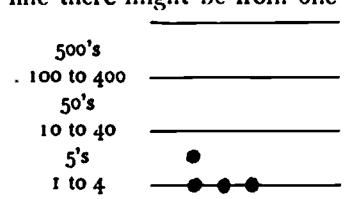


Japanese abacus

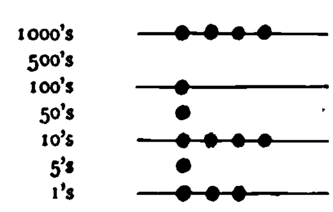
In Russia there is still used a type of abacus known as the *s'choty*, and not long ago a similar one was used in Turkey (the *coulba*) and also in Armenia (the *choreb*).

Now, to come nearer home, you have often bought things over a "counter" in a store, but did you know that the "counter" tells part of the story of addition and subtraction? Let us see what this story is.

We have already seen that Roman numerals, I, V, X, and so on, were in common use in Europe for nearly two thousand years. It was difficult, however, to write large numbers with these numerals. For example, 98,549 might be written in this way: lxxxxviijMDXLVIIII. There were other ways of writing this number, but they were equally clumsy. The merchants therefore invented an easier method of expressing large numbers. They drew lines on a board, with spaces between the lines, and used disks (small circular pieces like checkers) to count with. On the lowest line there might be from one to four disks, each

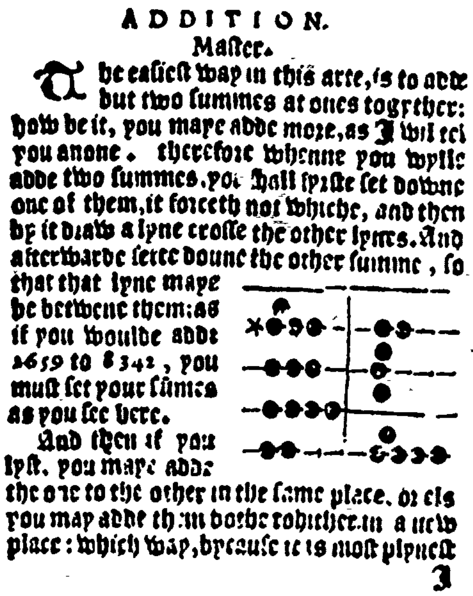


disk having the value of 1. A disk in the space above had the value of 5, and this combined with the disks on the line below could give 6, 7, 8, or 9. In this illustration $5 + 3$ is represented. Larger numbers were handled in the same way on the upper lines and spaces. Sometimes the counters were slid along the rods.



Now look at this figure. There are four disks on the thousand's line, none in the five hundred's space, and so on; that is, you have $4000 +$ no 500's $+ 100 + 50 + 40 + 5 + 3$, or 4198.

You may care to see how such a counting board, with its counters, looked in one of the oldest English arithmetics about four hundred years ago. Maybe you would like to read the old English words as they were printed in the book. The problem is to add 2659 and 8342. The two numbers are not written but are expressed by counters. The star is put on the board so that the eye may easily see where the thousands come, just as we write a comma in a number like 2,659 to show that the left-hand figure is 2 thousand. (This is usually omitted in a date like 1937, and in many other cases, especially where there are numbers of only four figures.)



ADDITION.

Master.

The easiest way in this arte, is to adde but two summes at ones togyther: how be it, you maye adde more, as I wil tel you anone. therefore whenne you wylle adde two summes, you shall fyrste set downe one of them, it forceth not whiche, and then by it draw a lyne crosse the other lynes. And afterwarde sette doune the other summe, so that that lyne maye be betwene them: as if you woulde adde 2659 to 8342, you must set your summes as you see here.

And then if you lyst, you maye adde the one to the other in the same place, or els you may adde them bothe togither in a new place: which way, bycause it is most playnest

I

Page from Robert Record's *Ground of Artes*, printed nearly four hundred years ago

Because these disks were used in counting they were called *counters*, and the board was sometimes called a *counter board*. When the European countries gave up using counters of this kind (quite generally four hundred years ago) they called the boards

used in the shops and banks "counters," and this name has since been commonly used for the bench on which goods are shown in stores. The expression "counting house" is still used in some places to designate the room in which accounts are kept.

One reason for using the counters was that paper was not generally known in Europe until about the eleventh century. Boards covered with a thin coat of wax had been used from the time of the Greeks and Romans, more than a thousand years before. On these it was possible to scratch numbers and words, erasing them by smoothing the wax with a spoon-shaped eraser, but it was very slow work. Slates were used in some parts of Europe, but usable slate quarries were not common and therefore slates could not readily be used elsewhere.

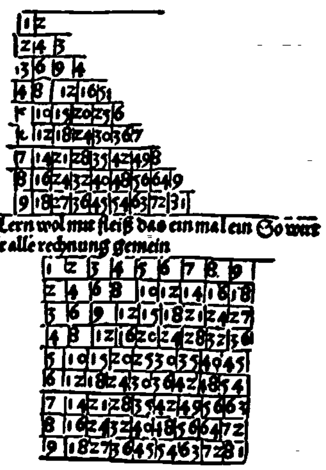
When blackboards were first made, chalk was not always easily found, and so written addition was not so common as addition by counters. When slates, blackboards, and paper all came into use, people added about as we do now. Since all careful computers "check" their work by adding from the bottom up and then from the top down so as to find any mistakes, pupils today add both ways, and there is no reason for teaching addition in only one direction.

Subtraction was done on the counting boards in much the same way as addition. The numbers were represented by counters and were taken away as the problem required. The terms "carry" and "borrow" had more meaning than at present, because a counter was actually lifted up and carried to the next place. If one was borrowed from the next place, it was actually paid back.

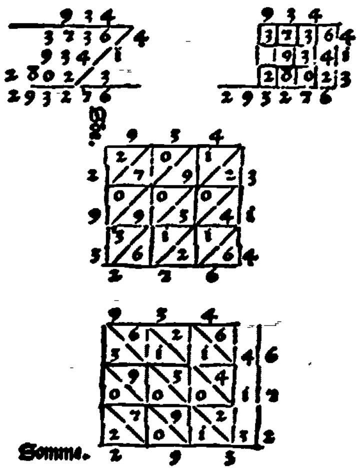
Today we learn the multiplication facts just as we learn to read words. If we need to use 7×8, we simply think "56," just as we think "cat" when we see the word CAT. Formerly, however, the "multiplication table" was first written down and then learned as a whole. On the following page are two of these tables from one of the oldest printed arithmetics, a German book of 1489 by Johann Widman. You may wish to see how they were arranged and how to find, in each table, the product of 8×9.

You may also like to see how multiplication looked in 1478, and

to find the meaning of the four cases on the right. It will be easier for you if you are told that in each one the problem is to multiply 934 by 314, the product being 293,276, the comma not being used at that time. Do you see how it was found?

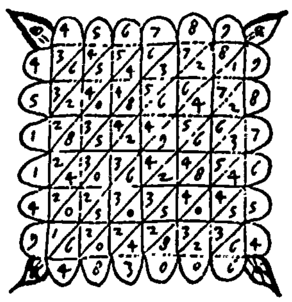


From Johann Widman's Arithmetic of 1489



From an arithmetic printed in Treviso, Italy, in 1478

Here is another example of multiplication reproduced from a manuscript of the fifteenth century showing how the product of 456,789 × 987,654 was found. The product of these figures is here shown to be 451,149,483,006. The picture looks something like a grating, and so this was sometimes called the "grating method." You may care to see if you can multiply 345 by 678 using this method. You may also care to try multiplying a 4-figure number by (say) a 3-figure number. If you understand the above illustration, it is not difficult to make such multiplication.



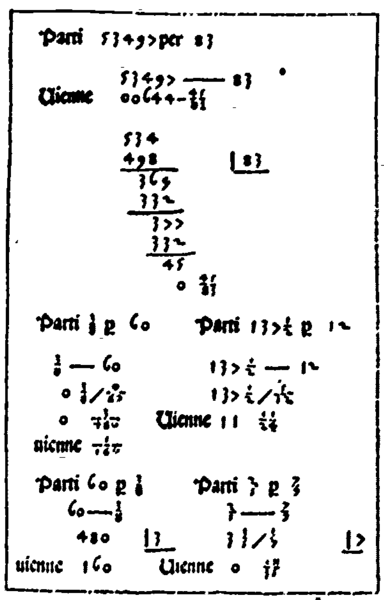
Division was rarely used in ancient times except where the divisor was very small. Indeed, at the present time it is not often needed in comparison with multiplication, and it is far more rarely employed than addition and subtraction. On the abacus it was often done by subtraction; that is, to find how many times 37 is contained in 74, we see that $74 - 37 = 37$, and $37 - 37 = 0$, so that 37 is contained twice in 74.

Here is another way of dividing that was the most common of any in the fifteenth century. It shows the division of 1728 by 144. As fast as the numbers had been used they were scratched out, and so this was often called the "scratch method." The number 1728 was first written; then 144 was written below it. Since the first figure in the quotient is 1, the numbers 144 and 172 are scratched out, and 144 was again written below. The remainders are written above. Divide 1728 by 144 as you would do it today, and compare your method with this.

```
  2
 38
~~1728~~(12
~~1444~~
 ~~14~~
```

Our present method, often called "long division," began to be used in the fifteenth century. It first appeared in print in Calandri's arithmetic, published in Florence, Italy, in 1491, a year before Columbus discovered America. The first example shown gives the division of 53,497 by 83, the result being $644\frac{45}{83}$. The second example is the division of $\frac{3}{8}$ by 60, the result being $\frac{1}{160}$. The other three are

$$137\tfrac{1}{2} \div 12 = 11\tfrac{11}{24}$$
$$60 \div \tfrac{3}{8} = 160$$
$$\tfrac{2}{5} \div \tfrac{7}{9} = \tfrac{18}{45} \div 7 = \tfrac{18}{35}$$



You may be interested in following each step in these examples from this very old arithmetic.

This interesting picture below is from a book that was well known about four hundred years ago. The book was first printed in 1503 and it shows two styles of computing at that time—the counters and the numerals. The number on the counting board at the right is 1241. The one at the left represents the attempted division of 1234 by 97, unsuccessful because decimal fractions such as we know were not yet invented.

From the *Margarita Philosophica*, by Gregorius Reisch

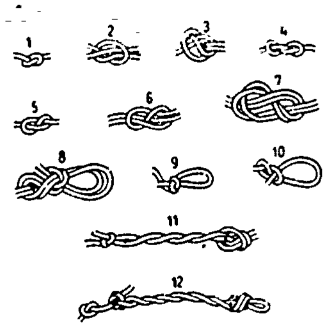
A method of computing by counters on a board seems to have been invented by the Indians of South America before the advent of the European discoverers, although our first evidence of the fact appeared in a Spanish work of 1590. In this a Jesuit priest, Joseph de Acosta, tells us:

"In order to effect a very difficult computation for which an able calculator would require pen and ink . . . these Indians [of Peru] made use of their kernels [*sus granos*] of wheat. They place one here, three somewhere else and eight I know not where. They move one kernel here and three there and the fact is that they are able to complete their computation without making the smallest mistake. As a matter of fact, they are better at calculating what each one is due to pay or give than we should be with pen and ink."*

In South America, apparently long before the European conquerors arrived, the natives of Peru and other countries used knotted cords for keeping accounts. These were called *quipus*, and were used to record the results found on the counting table. How old this use of the *quipu* may be, together with some kind of abacus, we do not know. A manuscript written in Spanish by a Peruvian Indian, Don Felipe Huaman Poma de Ayala, between 1583 and 1613 has recently been discovered, and is now in the Royal Library at Copenhagen. It contains a large number of pen-and-ink sketches. One of these is here reproduced from a booklet published in 1931.* This portrays the accountant and treasurer (*Cōtador maior i tezorero*) of the Inca, holding a *quipu*. In the lower left-hand corner is a counting board with counters and holes for pebbles or kernels.

* Henry Wassén, *The Ancient Peruvian Abacus*, Göteborg, 1931; reprinted from *Comparative Ethnographical Studies*, vol. 9.

The use of knots to designate numbers is found in Germany in connection with the number of measures of grain in a sack Menninger shows the shapes of the knots and their numerical meaning as follows:



In recent times machines have been invented in Europe and America by which we can perform all the operations with numbers far more rapidly than is possible with the abacus or with pencil and paper. You have often seen the cash register used in stores, and this adds very quickly and accurately. You may also have seen machines for making change, in which, for example, you place a dollar bill to pay for something that costs 37 cents. The salesman simply pushes a button marked $1 and another one or two to represent 37¢, and the change (63¢) comes out.

The time will come when high schools giving commercial courses not only will have classes in stenography and typewriting, but will give instruction in the use of business machines of various kinds, including calculators. The simplest types of calculators merely give results in addition and subtraction. Others list numbers, add, subtract, multiply, and divide. Bookkeepers no longer need to figure percentages; machines do this work more rapidly than any person can, and with less risk of errors. Many types of these calculators can be operated by electricity, and some are so small that they can be easily carried about by hand.*

* Calculating machines of many kinds are described in an excellent work by Perley Morse, *Business Machines*, Longmans, Green and Co., 1932.

V. Fractions

IN ANCIENT times it was harder to learn how to use fractions than it is today, because then no one knew any easy way to write them. About 1700 B.C. the Egyptians tried to invent an improved system of writing fractions. Employing our own numerals, so that you can understand them better, let us see how teachers in the Egyptian temple schools wrote the fraction 2/43. It was done as follows:

1/42, 1/86, 1/129, 1/301

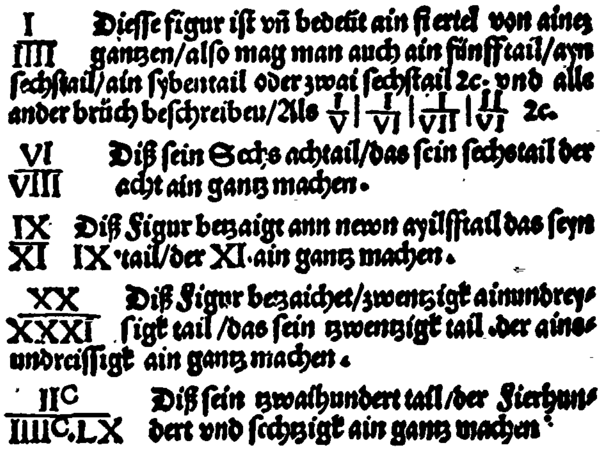
That is, the sum of these unit fractions (numerators all 1's) equals 2/43. These unit fractions were used for many centuries and in many countries, and as late as the seventeenth century certain Russian documents speak of a "half-half-half-half-half-third" instead of 1/96.

The ancient Greeks often used marks somewhat like our accents to represent unit fractions. Just as if we should write 4″ for 1/4, so they wrote Δ″, the Δ being their symbol for 4.

In general, however, the ancients avoided fractions by giving special names to parts of various measures. Indeed, we do the same today. Instead of saying "1 foot and five-twelfths," we say "1 foot and 5 inches," writing this as 1 ft. 5 in. In the same way peoples generally, for many centuries, have used such numbers as 7 mi. 50 ft.; 3 lb. 8 oz.; 5 sq. ft. 12 sq. in., just as we use them today.

About four hundred years ago a German arithmetic (Köbel's, 1514) used Roman numerals in writing fractions. These are seen in the illustration on page 36. The last of the fractions is 200/460, but these forms were not commonly used.

The astronomers avoided the difficulty by using degrees, minutes, and seconds, as in 45° 30′ 4″ (45 degrees, 30 minutes, four seconds). This usage has continued until today, and is seen in our writing of hours, minutes, and seconds, as in 3 hr. 10 min. 30 sec., equivalent to 3 hr. $\frac{1}{6}$hr. and 1/120 hr., or 3 and 21/120



I/IIII Dieſſe figur iſt vñ bedeũt ain fiertel von ainez ganzen/alſo mag man auch ain fünfftail/ayn ſechſtail/ain ſybentail oder zwai ſechſtail zc. vnd alle ander brüch beſchreiben/Als $\frac{I}{V}$ | $\frac{I}{VI}$ | $\frac{I}{VII}$ | $\frac{II}{VI}$ zc.

VI/VIII Diß ſein Sechs achtail/das ſein ſechstail der acht ain ganz machen.

IX/XI Diß Figur bezaigt ann newn ayilfftail das ſeyn IX tail/der XI ain ganz machen.

XX/XXXI Diß Figur bezaichet/zwenzigt ainundrey ſigt tail/das ſein zwenzigt tail der ainundraiſſigt ain ganz machen.

IIC/IIIIC.LX Diß ſein zwalhundert tail/der Sierhundert vnd ſechzigt ain ganz machen.

A page from Jakob Köbel's arithmetic (1514)—edition of 1564

hr. or 3 $\frac{7}{60}$ hr. We see that these, and also such numbers as 2 ft. 6 in., are really both measures and fractions.

We therefore have four kinds of fractions which we use even today.

1. The astronomical fractions, like 15° 15′ or 15 hr. 15 min.
2. Measurement fractions, like 2 ft. 3 in. instead of $2\frac{1}{4}$ ft.
3. Common fractions, so called because they were once more commonly used than the other two.
4. Decimal fractions, as in $2.75 instead of $$2\frac{3}{4}$. These are usually spoken of merely as "decimals."

Decimal fractions were introduced early in the sixteenth century, although some trace of their use is found earlier. In 1530 a German writer, Christoff Rudolff, published an arithmetic with some work on decimals, a bar being used instead of a decimal point, as in 393|75 for 393.75. Indeed, we have not even yet decided upon the best kind of decimal point. England in part of the eighteenth century used a period, as in 2.75, but later used 2·75 as at present. On the continent of Europe the comma is generally seen, as in 3,25. In the United States and Canada the period is used, as in $32.50.

The illustration here given shows how decimals were written in the Netherlands in 1585. The decimal fraction is now much more commonly used than the common fraction, but at one time it was not, and so we still use the name "common fraction." In Great Britain the term "vulgar fraction" is used, the Latin word *vulgus* meaning "common" and being adopted from the Latin books used in the schools of the sixteenth century.

Explication du donné. Il y a trois ordres de nombres de Diſme, deſquels le premier 27 ⓪ 8 ① 4 ② 7 ③, le deuxieſme 37 ⓪ 8 ① 7 ② 5 ③, le troiſieſme 875 ⓪ 7 ① 8 ② 2 ③.

Explication du requis. Il nous faut trouver leur ſomme. *Construction.* On mettra les nombres donnez en ordre comme ci joignant, les aiouſtant ſelon la vulgaire maniere d'aiouſter nombres entiers, en ceſte ſorte:

	⓪	①	②	③
2	7	8	4	7
3	7	6	7	5
8 7	5	7	8	2
9 4	1	3	0	4

From Simon Stevin's work—the first book ever published on decimal fractions

The abbreviations used in fractional parts of measures are often of interest. For example, in the British system of money we see 5£ 6*s*. 9*d*., or £5 6/9, meaning 5 pounds, 6 shillings, and 9 pence. £ is an abbreviation of the old Latin word *libra* (plural, *librae*), originally a pound of silver. The *s* originally stood for the Latin word *soldi* or *solidi*, a solidus being a solid piece of silver weighing $\frac{1}{12}$ of a pound. The old form of *s* was *ſ*, and this has of late been changed to / as in 6/9, "six shillings and nine pence." The *d* stands for the Latin *denarius*, originally 0.1 (but now 1/12) of a shilling and is thought of as "penny" (plural "pence").

We use the slanting line (/) today in writing common fractions, particularly on the typewriter and in print, since it takes up less room in print and is more easily written on the machine. Thus we write 5/8 for five-eighths, but this has no relation to its use in 6/9 as explained above.

By consulting a large dictionary you may find some interesting

facts relating to such words as mile, rod, yard, foot, and inch, as well as bushel, peck, gallon, quart, ounce, fathom, and knot. As to our dollar sign, formerly $, but now commonly written $, it came into use soon after 1800, but it was not generally recognized until about 1825 or 1830.

The metric system originated in France about 1800, but was not widely used until the middle of the century. It is now the standard in most of the leading countries, except that in the English-speaking ones it is not generally used in marketing and making other ordinary purchases. It is used in practically all science laboratories, and is legal in other lines. It is far simpler than our systems of measures (including the measures of weight), and is easily learned. The principal measures and their approximate equivalents are only these:

Length

meter (m.) = 39.37 in.
kilometer (km.) 1000 m. = 0.62 mi.
millimeter (mm.) 0.001 m. = 0.04 in.
centimeter (cm.) 0.01 m. = 0.4 in.

Remember that *milli* means 0.001, just as 1 mill = $0.001; *centi* means 0.01, just as 1 cent = $0.01; *kilo* means 1000.

Weight

gram = 0.03 oz.
kilogram = 2.2 lb.

Capacity

liter = 1 qt.

These measures should be learned by us all. If we need to use other metric measures, we can easily learn them when the time comes. It will be seen that in the metric system we make use of decimal fractions just as we do when we use dollars and cents.

Old-time fractions like 11/17, 37/117, and 131/1763 are now useless, and aside from such common fractions as halves, thirds, fourths, eighths, sixteenths, and thirty-seconds, the decimal fraction is the only one which we need to use in ordinary daily life.

VI. Mystery of Numbers

THERE are always in this world people who see mysteries in everything. Some of them pretend to tell fortunes by cards, sticks, tea leaves, dice, the hand, numbers, and other objects. Many think, or say they think, that the new moon should be seen over the right shoulder, that it is unlucky to walk under a ladder, or to spill salt, or to see a black cat crossing the road. Of course all this is childish nonsense, most of it probably coming down to us from ignorant tribes of people who could not read or write or even think straight about anything.

Among these superstitions is one which may have come from a curious use, made by various ancient peoples, of the numerical values so often given to the letters of the alphabet. These numerical values may be arranged in many ways. One of these ways is described in Chapter III. It consists in taking the first nine letters to represent the figures 1 to 9; the next nine to represent the tens; and the next to represent hundreds. The thousands are often indicated by a comma at the left of the hundreds.

This finding of values of words by the values of the letters is called *gematria*, a word of uncertain origin. Some writers believe that it is from the Greek *geometria* (geometry), and others that it comes from *gramma* (letter) or *grammateus* (a scribe or writer), or from a Hebrew word for "letters."

In some of the ancient alphabets there were not twenty-seven letters, in which case some that were found in still older alphabets were used to fill the list of numerals.

More than two thousand years ago, superstitious people believed that they could find the value of a person's ability by taking the value of the letters in his name. For example, take the names of John and Bill, by the scheme of letter values shown on page 12. You then have

J O H N	B I L L
$10 + 70 + 8 + 50 = 138$	$2 + 10 + 30 + 30 = 72$

so that John would excel Bill in wrestling, tennis, baseball, and so on. Of course Bill might have brains enough to spell his name Will, or William, which gives us:

W I L L W I L L I A M
$400+10+30+30=470$ $400+10+30+30+10+1+40=521$,

where we have given the Greek values to the letters, except that there being no Greek letter corresponding to 400, we have given to W the value for υ (the Greek *upsilon*). In either case John is thoroughly beaten, all of which goes to show the silliness of any such scheme.

We have, even today, people who believe in this ancient theory as amplified—a very easy matter—in what is known as numerology. The device was often used more legitimately in ancient times as a feature of cryptology (hidden writing). Thus we find in the Bible, in the book of Revelations, that 666 is "the number of the beast," used to represent some name known to the early Christians but now lost. There is reason to believe that it referred to "Nero Caesar," which name has this value when written in the Hebrew alphabet. In much later times it has often been used to refer to some prominent man who was disliked by the writer. Thus it has come about that the "number of the beast" has been assigned by their enemies to Luther, to various popes, to Mohammed, to Gladstone, and to others of high rank.

To look at the number values from another point of view, there are instances of tombstones bearing not only the name of the deceased but also its corresponding number. An even more curious use of the number-letter is found in certain Christian manuscripts where the number 99 is written at the close of a prayer. This is because, in the Greek alphabet, the letters of the word "Amen" give this value. These letters are A (1) + M (40) + H (8) + N (50), making a total of 99. (In Greek there are two *e*'s, *epsilon* and *eta*, the latter resembling our *h* and being the long form *eta*.)

The various classes of numbers also have their own stories, a few of which will now be considered. About as far back as we can go in history the belief that "there is luck in odd numbers"

seems to have been common. The odd numbers were looked upon as masculine and heavenly; the even numbers were feminine and earthly. The number 13 was an exception among odd numbers, having been looked upon as unlucky long before the Christian era. It is very strange that people, supposed to be highly civilized, avoid this number even today, as in the thirteenth floor, thirteen at a table, or compartment 13 in a sleeping car.

As to odd numbers in general, Shakespeare remarks, in *The Merry Wives of Windsor*, that "there is divinity in odd numbers, either in nativity, chance, or death." Even Plato, best known of all the ancient Greek scholars, wrote: "The gods below . . . should receive everything in even numbers, and of the second choice, and ill omen; while the odd numbers, and of the first choice, and the things of lucky omen, are given to the gods above."

As we have already learned, what we call the number one, the unit (from the Latin word *unus*, meaning one), was not generally considered a number in ancient times. It was looked upon as the basis of counting, as when we say that the inch is a unit of measure, or the mile a unit for long distances; that is, we count by inches, miles, pounds, hours, and so on. For this reason the early Latin writers said that it was the *fons et origo numerorum* (the fount and origin of numbers). Therefore two was generally looked upon as the first number in counting.

When the ancients spoke of the theory of numbers they referred to positive integers (whole numbers). The late Roman writers divided the numbers as digits (fingers, *digiti*), articles (joints, *articuli*), and composites (*compositi*) of fingers and joints:

digits, (1), 2, 3, . . ., 9
articles, joints, 10, 20, 30 . . ., 90
composites, numbers like 15, 27, 39, . . ., 99

The number 1 was at times omitted. After about the year 1000, figures from 1 to 9 were called "significant figures" to distinguish them from zero (0), although the 0 is often very significant.

In early times all the numbers from 1 to 12 had their mystic relations to religion and daily life. The special ones of importance were 3, 5, and 7, these being the prime numbers (not divisible by

any number except 1 and the number itself). We therefore have "three cheers," the trident (three-tooth)—the scepter of Neptune—and the Trinity, and many cases of the use of the number three in the Bible and in the ceremonies of the Church.

In China and other parts of the East 5 was particularly esteemed, but in Europe it was not considered so mysterious as 3 and 7, probably because it was half of 10, the root (radix) of our system of counting, and itself a radix in some parts of the world. Pythagoras, the Greek mathematician, who lived about 500 B. C., used the 5-pointed star as the symbol of his order. He may have brought it from the East, which he possibly visited.

Aside from 3, the most popular of the mystic numbers was 7. Thus the ancients spoke of the "seven wonders of the world," the "seven wise men of Greece," the "seven liberal arts" (arithmetic, geometry, astronomy, music, grammar, dialectics, and rhetoric), and the "seven planets," and we have the "seven days" of the week. The literature of the number 7 is extensive.

The number 9 was feared by the followers of Pythagoras, a superstition extending from about 500 B.C. to relatively modern times. It was thought to be a symbol of misfortune. There were many instances, however, in which 9 was used in other ways, as in the case of the Nine Muses. Among the Hebrews on the Day of Atonement nine calves were sacrificed, and Dante speaks of nine circles in his *Inferno*.

The learned Rabbi ben Ezra, well known to English readers from Browning's poem, devised this little diagram. It gives the multiplication table of 9's in a curious form.

	9	—— 9 = 9 × 1	
9 × 9 = 81	8	⇄ 1	18 = 9 × 2
9 × 8 = 72	7	⇄ 2	27 = 9 × 3
9 × 7 = 63	6	⇄ 3	36 = 9 × 4
9 × 6 = 54	5	⇄ 4	45 = 9 × 5

The cabala (or cabbala) was a mystical tradition handed down by the Jews from generation to generation by word of mouth. The Hebrew name is *qabbalah*, tradition. Among these traditions is one that shows that God is Truth, for the Hebrew word for Truth is *Em-th*, and in the Hebrew alphabet the numerical values of the letters add to 441, and $4 + 4 + 1 = 9$, the number which, as illustrated below, shows the invariance of God; for

$$9 \times 1 = 9, \text{ and } 0 + 9 = 9$$
$$9 \times 2 = 18, \text{ and } 1 + 8 = 9$$
$$9 \times 3 = 27, \text{ and } 2 + 7 = 9$$
$$\vdots$$
$$9 \times 9 = 81, \text{ and } 8 + 1 = 9$$

The following i· other tradition of the cabalists: They assert that the Seal of God is Truth, the *Em-th* referred to above. Taking the Hebrew name for Jehovah, only the four letters J, H, V, and H were used in writing this name. The values in the Hebrew alphabet add up to 26, and $2 + 6 = 8$. The value of the Holy Name (26) repeated twice is then 2×26, or 52, and $5 + 2 = 7$. Therefore

$1 \times$ the Holy Name = 26, with $2 + 6 = 8$
$2 \times$ the Holy Name = 52, with $5 + 2 = 7$
$3 \times$ the Holy Name = 78, with $7 + 8 = 15$ and $1 + 5 = 6$
.
.
.
8 = 208, with $2 + 0 + 8 = 10$, and $1 + 0 = 1$
9 = 234, with $2 + 3 + 4 = 9$

The moral which the cabalists drew from this was that the more godliness there is in a man, the less he thinks of himself, 9 being perfect godliness. Men who had nothing of importance to do wasted a great deal of time in finding such number relations.

VII. Number Pleasantries

For thousands of years people have played with numbers, and many of the puzzles and curious problems that they have invented have come down to us. Some of these were invented in Egypt at least four thousand years ago; others came from India, and from Greece; while many have come to us from Rome and from what were once her provinces.

It would take a long time to tell all the story of these puzzles and problems; but it is possible to give in this book a few of the number pleasantries that have developed in modern times.

If you become really interested in numbers, you will find many things that will seem very strange. Here is a curious case relating to 45. It is not very difficult to see why the statement must be true.

$$\begin{array}{r} 987,654,321 \\ \underline{123,456,789} \\ 864,197,532 \end{array}$$

If you take the number 987,654,321, made up of the nine digits, reverse it, and subtract like this, you will have three numbers—the minuend, the subtrahend, and the remainder—and the sum of the digits of each of the three is exactly 45.

Here is another curious puzzle: Show how to write one hundred, using only the nine figures from 1 to 9, and the signs of arithmetic. This illustration shows how it may be done.

$$100 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9$$

Here is still another interesting case. You can probably see what 18×37 must be, and what is the sum of the digits in the product. If so, you can tell the same for 21×37 and 27×37.

$$\begin{array}{rl} 3 \times 37 = 111, & \text{and}\quad 1 + 1 + 1 = 3 \\ 6 \times 37 = 222, & \text{and}\quad 2 + 2 + 2 = 6 \\ 9 \times 37 = 333, & \text{and}\quad 3 + 3 + 3 = 9 \\ 12 \times 37 = 444, & \text{and}\quad 4 + 4 + 4 = 12 \\ 15 \times 37 = 555, & \text{and}\quad 5 + 5 + 5 = 15 \end{array}$$

You may care to look at the first curious set of multiplications below. It will probably be not at all difficult for you to give

the products of 15,873 by 28, 35, 42, and 49. In the same way, after studying this case, beginning with $0 \times 9 + 8$, and $9 \times 9 + 7$, you can carry the work still farther, to $98,765 \times 9 + 3$, and so on, as shown at the right.

$$\begin{array}{r}
7 \times 15{,}873 = 111{,}111 \\
14 \times 15{,}873 = 222{,}222 \\
21 \times 15{,}873 = 333{,}333
\end{array}
\qquad
\begin{array}{r}
0 \times 9 + 8 = 8 \\
9 \times 9 + 7 = 88 \\
98 \times 9 + 6 = 888 \\
987 \times 9 + 5 = 8888 \\
9876 \times 9 + 4 = 88888
\end{array}$$

Here is another step in the endless game of numbers. You can easily see what the result of $123{,}456 \times 8 + 6$ must be, and you may care to see how far you can go in this series of numbers, and to find out why you cannot go farther. The same may be done with the arrangement below. If you are interested in these curious number games, see if you can work out a new one for yourselves.

$$\begin{array}{r}
1 \times 8 + 1 = 9 \\
12 \times 8 + 2 = 98 \\
123 \times 8 + 3 = 987 \\
1234 \times 8 + 4 = 9876 \\
12345 \times 8 + 5 = 98765
\end{array}$$

$$\begin{array}{r}
1 \times 9 + 2 = 11 \\
12 \times 9 + 3 = 111 \\
123 \times 9 + 4 = 1111 \\
1234 \times 9 + 5 = 11111 \\
12345 \times 9 + 6 = 111111 \\
123456 \times 9 + 7 = 1111111 \\
1234567 \times 9 + 8 = 11111111 \\
12345678 \times 9 + 9 = 111111111
\end{array}$$

There is another interesting lot of games of which this is an illustration. Find the figures (numerals) to put in place of these letters so as to make the multiplications exact. It may help you to know that C is 4 and T is 8.

$$\begin{array}{cccc}
 & I & C & C \\
 & & I & N \\
\hline
 & N & T & T \\
I & C & C & \\
\hline
I & A & N & T
\end{array}
\qquad
\begin{array}{cccc}
 & I & N & U \\
 & & N & U \\
\hline
 & L & N & U \\
N & U & S & \\
\hline
O & I & N & U
\end{array}$$

Here is another case. Just to give you a start, the word *emulations* has ten letters. Possibly you can find a similar word beginning with IN that will help you in the preceding multiplications. It

happens that both end in S, and maybe that stands for zero (0)

```
        E M A                    M T
M A )U E M A          E M A )U U S S
     M A                     M A S
       T M                     O S S
       A S                     I A S
       E M A                     A S
       E M A
```

You can easily make up others by first finding a word of ten letters in which no letter is repeated.

The question is sometimes asked in school, "What is the largest number?" The answer is, "There is no such thing, for however large a number may be mentioned, by adding 1 to it a larger number is found." It is like asking, "What is the longest time?" or "What is the greatest distance in space?" We can write very large numbers, however, but we do not need to do it in the ordinary way. We must first know what certain numbers mean:

3^2 is read "three square," and it means 3×3, or 9;
3^3 is read "three cube," or "three to the third power," and it means $3 \times 3 \times 3$, or 27;
3^4 means $3 \times 3 \times 3 \times 3$, or 81.
3^{4^2} means $3^{4 \times 4}$ or 3^{16} or 16 threes multiplied together, or 43,046,721.

The greatest of the ancient Greek mathematicians was Archimedes, who lived about 250 B.C. In his time the largest number for which the Greeks had a name was a *myriad*, or 10,000. They could speak of larger numbers, however, for they could say "a myriad myriads," which would be $10{,}000 \times 10{,}000$, or 100,000,000, our "hundred millions." Numbers from 1 up to this point were "numbers of the first *order*"—that is, numbers up to 10^8.

Numbers between 10^8 and 10^{16} were said to be of the "second order," and those between 10^{16} and 10^{24} were of the "third order." If we should write 10^{24} in our ordinary way we would have 1 with 24 zeros after it. If we should go on until we reached a number of the 10,000,000th order we would have 1 with 80,000,000 zeros.

We call numbers up to this point "numbers of the first *period*."

If we should now keep on through the "second period," the "third period," and so on to the 1,000,000th period, the first of this period would be 1 with 80,000 billions of zeros after it. This number would be far beyond our imagination, but it is small when compared with the greatest number expressed by four figures. This is

$$9^{9^{9^{9}}} \text{ or } 9^{\left(9^{9^{9}}\right)}$$

First we should understand that the ninth power of 9, or 9^9, means

$$9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$$

and this is easily found to be 387,420,489. The number 9 must now be raised to this power; that is, we must find the value of $9^{387,420,489}$. We do not know what the result would be, but we do know that there would be about 300,000,000 figures in it, the first twenty-seven being 428,124,773,175,747,048,036,987,115, and the last two being 89, but what comes between these two we do not know. If the number were written on a strip of paper in figures large enough to be easily read, the strip would be somewhere between 740 miles and 1100 miles long. To print it would require 33 volumes of 800 pages each, each page containing 1400 figures.

But this is only a beginning. The number 9 would again have to be raised to the power indicated by the number above referred to as having about 300,000,000 figures. The result would be what the great German mathematician Gauss characterized as "measurable infinity."

To get some further idea of its immensity we may consider that

$$(9^9)^{9^9} = 387,420,489^{387,420,489}$$

would lead to an enormous number, but it would be insignificant in comparison with the value of $9^{9^{9^{9}}}$.

If, therefore, we see that even a number written with only four figures, three being exponents, has a value that could not possibly be written by any human being, we may say that "there is no such thing as the greatest number." Like endless space it is too immense to be imagined.*

* W. Lietzmann, *Riesen und Zwerge im Zahlenreiche*. p. 32. Leipzig, 1932.

VIII. Story of a Few Arithmetic Words

IF YOU were interested in the story of number names as told in Chapter II, you will be even more interested in the story of some of the other words as used in arithmetic. Some of these words are very old, and some of them will in due time be forgotten.

Addition. Latin *ad* (to) + *dare* (put). To put numbers together. From this comes *addend*, Latin *addendum*, something to be added.

Subtraction. Latin *sub* (below) + *trahere, tract* (draw). Drawing one number down from another.

Multiplication. Latin *multus* (many) + *plicare* (fold); therefore originally the same as manifold.

Division. Latin *dis-* (apart), related to *duo* (two) + *videre* (see). See in two parts; more generally, to separate int a certain number of parts.

Sum. Latin *summus* and *summa* (highest). Formerly used in multiplication as well as in addition, and also to represent any arithmetical problem, as "do these sums."

Difference. Latin *dis-* (apart) + *ferre* (bear, carry). Result of taking away a part of a number.

Product. Latin *pro-* (in front of) + *ducere* (lead). Carry forward. It has been used for the result of an addition, but is now used generally in multiplication.

Minuend. From Latin *minor* (and *minus*, less), hence to make less.

Quotient. Latin *quotiens* (how many times). The number showing how many times one number is contained in another.

Root. Latin (related to) *radix* (root), the modern *radish.* Some of our words came from the Arabs of about the ninth century. They thought of a square number as growing from a root, like a tree. Their word for root was translated into Latin, whence *radix.* The Latins thought of a square root as the side (*latus*) of a square, and so they spoke of "finding" a latus, whereas the Arabs thought of "pulling out" a radish (root), whence *ex-* (out) + *trahere, tractum*

(pull, draw) and a trace (drawing), track, drag. The word *latus* is seen in lateral (sided) as in *quadrilateral*, Latin *quadri-* (four) + lateral.

Fraction. Latin *frangere, fractus* (to break, broken). The early English arithmetics often spoke of fractions as "broken numbers." A fractured bone is one that is broken. In England the fractions most commonly used in business, like $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{8}$, are called vulgar fractions (Latin *vulgus*, common people); but in America the word "vulgar" came to mean coarse or low, and the word "common" came into use in connection with fractions. As we have seen, there are other kinds of fractions, such as astronomical fractions (2° 10′), which are so called because they are used by astronomers in measuring angles between stars and for locating the heavenly bodies.

Numerator. Latin *numerus* (number), whence *numerator* (numberer). The number in a common fraction written above the horizontal ($\frac{3}{4}$) or before the slanting line (3/4). In $\frac{3}{4}$ the 4 names the fraction (that is, its family name is "fourths"). The 3 tells how many fourths there are; that is, the numberer (*numerator*) is 3.

Denominator is a Latin word meaning "namer." It is the number below the line in a common fraction. It names the kind of fraction, like fourths or eighths, and sometimes appears as *nominator* (namer) or *nomen* (name).

You will find it interesting to look in any large dictionary and find the original meaning of words like those given above. The uses of other numerical words may also be seen by consulting a dictionary. Here are a few:

One. Latin *unus* (one). Found in "uniform" (one form, alike) and many other words beginning with *uni-*, as in "unicorn," "unit," "unique," and "universe."

Two. Latin *duo* (two). Found in "duet," a piece of music for two, and many other words beginning with *du-*, as in "duplex," "dual," and "duplicate."

Three. Latin and Greek *tri-* (three). Found in "triangle" (three-angle) and many other words beginning with *tri-*, as in "trident" (three-teeth, the *dent* as being found in "dentist").

Four. An old English word, *feower;* German *vier.* Found in "fourfold," "fourteen" (4 + 10), and many similar words. The Latin is *quatuor* (four). Found in "quart," more directly from *quartus* (fourth), a fourth of a gallon, and in "quarter," and "quartet."

Five. An old English word, *fif;* German *fünf.* Found in "fifteen" (5 + 10), "five-fold," and other words. The Latin is *quinque* and Greek is *pente.* These appear in "quintessence," a fifth essential part; and in "quintet" in music; and in "pentagon" (five-angle, *gonia* being the Greek for "angle"); and in "pentateuch" and "pentecost."

Six. Latin *sex* and Greek *hex.* Found in "sextant," "sexagesimal," and "sexagenarian"; and in "hexagon" (six-angle) and "hexahedron" (six-face).

Seven. Latin *septem* (seven). Found in "September" (seventh month in the old calendar which began in March), and in "septet."

Eight. Latin and Greek *octo* (eight). Found in "octagon" (eight-angle), "octave" (stanza of eight lines), and October (eighth month in the old calendar).

Nine. Latin *novem* (nine). Found in "November" (ninth month in the old calendar). The English form (nine) is found in "nineteen" (9 + 10), and "ninety" (nine tens).

Ten. Latin *decem* (ten), Greek *deka.* Found in "December" (tenth month in the old calendar), "decade" (ten, often applied to ten years), "decagon" (ten-angle), and "decalogue" (ten words, the ten commandments of the Bible).

For you we have now opened the door to a room richly furnished—a large dictionary. In this room find the stories of other words used in arithmetic, including the words "arithmetic" itself, and "mathematics," and "school," and "cent," and "dollar."

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